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Darboux transformations for the nonlinear Schrödinger equations

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Abstract. Darboux transformations for the AKNS/ZS system are constructed in terms of Grammian-type determinants of vector solutions of the associated Lax pairs with an operator spectral parameter. A study of the reduction of the Darboux transformation for the nonlinear Schrödinger equations with standard and anomalous dispersion is presented. Two different families of new solutions for a given seed solution of the nonlinear Schrödinger equation are given, being one family related to a new vector Lax pair for it. In the first family and associated to diagonal matrices we present topological solutions, with different asymptotic argument for the amplitude and nonzero background. For the anomalous dispersion case they represent continuous deformations of the bright n-soliton solution, which is recovered for zero background. In particular these solutions contain the combination of multiple homoclinic orbits of the focusing nonlinear Schrödinger equation. Associated with Jordan blocks we find rational deformations of the just described solutions as well as pure rational solutions. The second family contains not only the solutions mentioned above but also broader classes of solutions. For example, in the standard dispersion case, we are able to obtain the dark soliton solutions.

1. Introduction

The nonlinear Schrödinger (NLS) equation is one of the more relevant among the set of integrable equations in $(1+1)$-dimensions and has been extensively studied since the seminal papers [14, 15, 1]. Its role in nonlinear optics is central to the study of solitons in optical fibres [9] and as a partial differential equation is a universal equation describing the propagation of a quasi-monochromatic wave in a weakly dispersive nonlinear one-dimensional media. In [6] one can find a very detailed study of the inverse scattering and the Riemann problem as well as its Hamiltonian structure.

Darboux transformations are one of the main tools in the theory of integrable systems [11]. Given a spectral problem defined for some potentials the Darboux transformation acts on the potentials and wavefunctions at the same time giving us solutions to a similar problem. When applied to the Lax pairs associated to integrable systems one obtains new solutions from old ones, an auto-Bäcklund transformation. For the Ablowitz–Kaup–Newell–Segur/Zakharov–Shabat (AKNS/ZS) these transformations have been analysed for example in [10], see also [11, 13, 5, 4].

Our results regarding the 3-waves resonant interaction equations [7] and the Davey–Stewartson (DS) equations and its Darboux transformation lead us to study the NLS equation from this point of view. We remark here that in [12] the Darboux transformations for the DSI were presented but the reduction to the NLS was just briefly considered. For the

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AKNS/ZS system we consider two adjoint Lax pairs, but now the wavefunctions are not scalar but vector functions on some complex linear space V and its dual V^*, and the spectral parameters now are arbitrary operators on V. This allows us construct new AKNS/ZS spectral problems with new potentials and wavefunctions. The advantages of this new simple approach are two-fold: first it allows us to obtain a compact way of obtaining multiple Darboux transformations by Grammian type determinants, secondly the reduction problem has at least two clear solutions. In fact, we have two different ways of reducing our Darboux transformations for the AKNS/ZS to the NLS equation in its focusing and defocusing cases. We must say that these two methods have their roots in the Darboux transformations for the DSI and DSII we arrive at in [8], respectively.

The first method is close to known results, having the advantage of getting limiting cases by considering the operator extension of the spectral parameter to be in its Jordan canonical form and not just a diagonal operator. This gives rational deformation of the bright soliton solutions of the focusing NLS.

The second method is completely new and is motivated by our previous results in the DSII equation [8]. In fact the construction allows us to obtain new formulae for the dark soliton solutions of the defocusing NLS equation. Until now the known Darboux transformations were unable to reproduce these solutions. Moreover, it contains also the bright soliton solution of the focusing case. This reduction provides us with a new Lax pair for the NLS equation.

The layout of the paper is as follows. In section 2 we present our results for the AKNS/ZS. Next, in section 3, we pay attention to the reduction problem and obtain theorems 2 and 3: the main results of the paper. Finally, the long section 4 is devoted to analysis of different examples of the two methods. Motivated by the importance of the nonzero background [3] we apply our two different approaches to it.

Within the first method we obtain topological deformations (different values for the argument of the amplitude) of the bright n-soliton solution. The deformation of the bright 1-soliton appears in [11]; however, the important topological aspect of it is missed. We also give some plots of this solution. As a special case one gets the combination of multiple homoclinic orbits for the focusing NLS [2]. Rational deformations of these topological deformations are presented as well, and are also pure rational solutions.

For the second method we present the general form of the dressing of the nonzero background. We take it as an example by obtaining the dark n-soliton solution of the defocusing NLS, to the author's knowledge this is the first time that these solutions have been obtain by using Darboux transformations. The formulae differ from the standard ones and are closer to those for the bright case.

2. Darboux transformations for the AKNS/ZS system

The AKNS/ZS equations for the complex functions $p(x,t), q(x,t)$, that depend on the complex variables $x, t \in \mathbb{C}$, are the following set of nonlinear partial differential equations:

$$\begin{aligned} \mathrm{i}\partial_t p &= \partial_x^2 p + 2p^2 q \\ \mathrm{i}\partial_t q &= -\partial_x^2 q - 2q^2 p \end{aligned} \tag{1}$$

where $\partial_t = \partial/\partial t$ and $\partial_x = \partial/\partial x$. It is well known [14, 1] that these equations are the compatibility conditions for the following linear system

$$\begin{aligned}
&\partial_x g_1 - L g_1 - q g_2 = 0\\
&\partial_x g_2 + L g_2 + p g_1 = 0\\
&\mathrm{i}\partial_t g_1 + 2L^2 g_1 + 2qL g_2 + pq g_1 + \partial_x q g_2 = 0\\
&\mathrm{i}\partial_t g_2 - 2L^2 g_2 - 2pL g_1 - pq g_2 + \partial_x p g_1 = 0.
\end{aligned} \tag{2}$$

In the standard formulation of this Lax pair the variable L is a spectral parameter in $\mathbb{C}$ and g_i are scalar complex wavefunctions. Nevertheless, if we replace $\mathbb{C}$ by an arbitrary complex linear space V, so that now the functions $g_i(x, t)$, $i = 1, 2$, take its values in V and $L \in \mathrm{L}(V)$ is an arbitrary linear operator, the compatibility condition for (2) is again the AKNS/ZS equations (1). Henceforth, we shall assume that our Lax pair is of vector character and that the spectral parameter is a general operator, not necessarily diagonal.

The adjoint Lax pair is

$$\begin{aligned}
&\partial_x \gamma_1 - \gamma_1 \Lambda - p\gamma_2 = 0\\
&\partial_x \gamma_2 + \gamma_2 \Lambda + q\gamma_1 = 0\\
&\mathrm{i}\partial_t \gamma_1 - 2\gamma_1 \Lambda^2 - 2p\gamma_2 \Lambda - pq\gamma_1 + \partial_x p\gamma_2 = 0\\
&\mathrm{i}\partial_t \gamma_2 + 2\gamma_2 \Lambda^2 + 2q\gamma_1 \Lambda + pq\gamma_2 - \partial_x q\gamma_1 = 0
\end{aligned} \tag{3}$$

here γ_i, $i = 1, 2$, takes its values in V^*, the dual of V, and therefore for each x, t they are linear functionals over V, and $\Lambda \in \mathrm{L}(V)$ is an arbitrary linear operator. System (3) is compatible if and only if p and q solve (1).

With the use of these vector wavefunctions g and γ we can construct the operator $\Phi(x, t)$ as follows:

Proposition 1. Given p, q, g_i, γ_i, L and Λ as above there exists locally an operator Φ solving the equations

$$\begin{aligned}
&\partial_x \Phi = g_1 \otimes \gamma_1 - g_2 \otimes \gamma_2\\
&\mathrm{i}\partial_t \Phi = 2[\partial_x(\Phi\Lambda - L\Phi) + p g_1 \otimes \gamma_2 - q g_2 \otimes \gamma_1]
\end{aligned} \tag{4}$$

such that

$$L\Phi + \Phi\Lambda = g_1 \otimes \gamma_1 + g_2 \otimes \gamma_2. \tag{5}$$

Proof. Using (2) and (3) one finds that

$$\begin{aligned}
\mathrm{i}\partial_t(g_1 \otimes \gamma_1 - g_2 \otimes \gamma_2) = {}& 2\partial_x((g_1 \otimes \gamma_1 - g_2 \otimes \gamma_2)\Lambda - L(g_1 \otimes \gamma_1 - g_2 \otimes \gamma_2)\\
&+ p g_1 \otimes \gamma_2 - q g_2 \otimes \gamma_1)
\end{aligned}$$

and therefore there exists a local potential Φ solving (4). Now, with the use of (2), (3) and (4) we immediately check that

$$\begin{aligned}
&\partial_x(L\Phi + \Phi\Lambda - g_1 \otimes \gamma_1 - g_2 \otimes \gamma_2) = 0\\
&\partial_t(L\Phi + \Phi\Lambda - g_1 \otimes \gamma_1 - g_2 \otimes \gamma_2) = 0
\end{aligned}$$

and hence if the initial condition $\Phi_0 = \Phi(x_0, t_0)$, that determines a unique solution of (4), is such that $(L\Phi + \Phi\Lambda - g_1 \otimes \gamma_1 - g_2 \otimes \gamma_2)|_{x_0, t_0} = 0$ (5) follows. □

We shall suppose that Φ is an invertible operator, and that it is Φ^{-1} which plays the role of a Darboux operator.

Proposition 2. The objects

$$\begin{aligned}\hat{p} &= p + 2\langle\gamma_1, \Phi^{-1}g_2\rangle\\ \hat{q} &= q + 2\langle\gamma_2, \Phi^{-1}g_1\rangle\\ \hat{g}_i &= \Phi^{-1}g_i\\ \hat{\gamma}_i &= \gamma_i\Phi^{-1}\\ \hat{L} &= -\Lambda\\ \hat{\Lambda} &= -L\end{aligned}$$

satisfy (2) and (3) if the objects $p, q, g_i, \gamma_i, L, \Lambda$ do.

Proof. Introduce $g_i = \Phi\hat{g}_i$ and $\gamma_i = \hat{\gamma}_i\Phi$ in (2) and (3) respectively. Then use the differential equations (4) for Φ and (5) to obtain the desired result. □

From this proposition it follows that

Theorem 1. Let p and q be a solution of (1) and g_i, γ_i solutions of (2) and (3) respectively, take an invertible operator Φ as in proposition 1, and define

$$\begin{aligned}\hat{p} &= p + 2\langle\gamma_1, \Phi^{-1}g_2\rangle\\ \hat{q} &= q + 2\langle\gamma_2, \Phi^{-1}g_1\rangle.\end{aligned}$$

Then $\hat{p}$ and $\hat{q}$ are new solutions of (1). Moreover, the following relation holds:

$$\hat{p}\hat{q} = pq + \partial_x^2 \ln\det\Phi. \tag{6}$$

Proof. From proposition 2 it follows that $\hat{p}$ and $\hat{q}$ are necessarily solutions of (1).

To prove the relation (6) we proceed as follows. First we observe that from proposition 1 one has the relations

$$\begin{aligned}\mathrm{Tr}(L + \Lambda) &= \langle\hat{\gamma}_1, g_1\rangle + \langle\hat{\gamma}_2, g_2\rangle\\ \partial_x \ln\det\Phi &= \langle\hat{\gamma}_1, g_1\rangle - \langle\hat{\gamma}_2, g_2\rangle.\end{aligned}$$

Where in the first relation we have use (5) and in the second (4) together with the identity $\mathrm{Tr}(\partial_x\Phi\Phi^{-1}) = \partial_x \ln\det\Phi$. Now, taking the x-derivative we find out

$$\begin{aligned}0 &= \partial_x\langle\hat{\gamma}_1, g_1\rangle + \partial_x\langle\hat{\gamma}_2, g_2\rangle\\ \partial_x^2 \ln\det\Phi &= \partial_x\langle\hat{\gamma}_1, g_1\rangle - \partial_x\langle\hat{\gamma}_2, g_2\rangle.\end{aligned}$$

We must remark that these two equations follow from (4), see the proof of proposition 1. So, we deduce

$$\partial_x^2 \ln\det\Phi = 2\partial_x\langle\hat{\gamma}_1, g_1\rangle = -2\partial_x\langle\hat{\gamma}_2, g_2\rangle.$$

We concentrate on the first relation. We evaluate

$$2\partial_x\langle\hat{\gamma}_1, g_1\rangle = 2\hat{p}\langle\hat{\gamma}_2, g_1\rangle + 2q\langle\hat{\gamma}_1, g_2\rangle = \hat{p}(\hat{q} - q) + q(\hat{p} - p) = \hat{p}\hat{q} - pq$$

were we have used (2) and (3) for both unhated and hated variables and the definitions of proposition 2. From this equation follows the desired result. □

We must remark that when L and Λ are diagonal $n \times n$ matrices these solutions might correspond to an iterated scalar Darboux transformation.

3. Darboux transformations for the NLS equations

When $x, t \in \mathbb{R}$ are real variables and $q = \varepsilon p^*$, henceforth in the paper $\varepsilon = \pm 1$, with p^* the complex conjugate of p, the AKNS/ZS system (1) reduces to

$$\mathrm{i}\partial_t p = \partial_x^2 p + 2\varepsilon |p|^2 p \tag{7}$$

which is the well known NLS equation in its focusing or anomalous dispersion ($\varepsilon = 1$) and defocusing or standard dispersion ($\varepsilon = -1$) cases.

The natural question that arises here is whether or not it is possible to reduce the Darboux transformations of section 2 to this equation. The surprising fact is that there are at least two different ways to give a positive answer to this question. We shall explain these two different constructions.

One of them needs V to be pre-Hilbert (i.e. with inner product), and so a map $^\dagger : V \to V^*$ is defined, and is based on the following observation.

Proposition 3. Given solutions g_1 and g_2 of (2) and a linear operator $H \in \mathrm{L}(V)$ with the interlacing property $L^\dagger H = H\Lambda$ the functionals $\gamma_1 = g_2^\dagger H$ and $\gamma_2 = \varepsilon g_2^\dagger H$, are solutions of (3) if and only if $q = \varepsilon p^*$.

Proof. From (2) one deduces for the defined γ's

$$\begin{aligned}
&\partial_x \gamma_1 - \gamma_1 \Lambda - \varepsilon q^* \gamma_2 = 0\\
&\partial_x \gamma_2 + \gamma_2 \Lambda + \varepsilon p^* \gamma_1 = 0\\
&\mathrm{i}\partial_t \gamma_1 - 2\gamma_1 \Lambda^2 - 2\varepsilon q^* \gamma_2 \Lambda - p^* q^* \gamma_1 - \varepsilon \partial_x q^* \gamma_2 = 0\\
&\mathrm{i}\partial_t \gamma_2 + 2\gamma_2 \Lambda^2 + 2\varepsilon p^* \gamma_1 \Lambda + pq\gamma_2 - \varepsilon \partial_x p^* \gamma_1 = 0
\end{aligned}$$

that when compared with (3) gives the result stated. □

How is this proposition of any help in our aim? First, if a solution p to the NLS equation (7) is given, we also have a solution $p, q = \varepsilon p^*$ of (1). Then, we perform a Darboux transformation to obtain a new solution $\hat{p}, \hat{q}$ of (1). To do that we need solutions g_1, g_2 of (2) and solutions γ_1, γ_2 of (3), these last two we take as prescribed in proposition 3. Then, the Darboux operator Φ is given by

$$\begin{aligned}
&\partial_x \Phi = (g_1 \otimes g_1^\dagger - \varepsilon g_2 \otimes g_2^\dagger) H\\
&\mathrm{i}\partial_t \Phi = 2[\partial_x(\Phi\Lambda - L\Phi) + \varepsilon(p g_1 \otimes g_2^\dagger - p^* g_2 \otimes g_1^\dagger) H]\\
&L\Phi + \Phi\Lambda = (g_1 \otimes g_1^\dagger + \varepsilon g_2 \otimes g_2^\dagger) H
\end{aligned}$$

and from the two first it follows that $H^\dagger \Phi - \Phi^\dagger H$ is a constant operator not depending on x or t, so that if the initial conditions are chosen in an appropriate way: $H^\dagger \Phi = \Phi^\dagger H$. With such a Φ we construct our Darboux transformation, the transformed wavefunctions are $\hat{g}_1 = \Phi^{-1} g_1$, $\hat{g}_2 = \Phi^{-1} g_2$ and

$$\begin{aligned}
&\hat{\gamma}_1 = \gamma_1 \Phi^{-1} = g_1^\dagger H \Phi^{-1} = g_1^\dagger (\Phi^\dagger)^{-1} H^\dagger = \hat{g}_1^\dagger \hat{H}\\
&\hat{\gamma}_2 = \gamma_2 \Phi^{-1} = \varepsilon g_2^\dagger H \Phi^{-1} = \varepsilon g_2^\dagger (\Phi^\dagger)^{-1} H^\dagger = \varepsilon \hat{g}_2^\dagger \hat{H}
\end{aligned}$$

here $\hat{H} = H^\dagger$, these last two solve (3) then proposition 3 shows that $\hat{q} = \varepsilon \hat{p}^*$ and the NLS reduction is preserved by the Darboux transformation.

So the first method is

Theorem 2. Given a solution p of the NLS equation (7) take functions g_1 and g_2, with values in a pre-Hilbert space V, solving the following linear system

$$\begin{aligned}
&\partial_x g_1 - Lg_1 - \varepsilon p^* g_2 = 0\\
&\partial_x g_2 + Lg_2 + pg_1 = 0\\
&\mathrm{i}\partial_t g_1 + 2L^2 g_1 + 2\varepsilon p^* L g_2 + \varepsilon |p|^2 g_1 + \varepsilon \partial_x p^* g_2 = 0\\
&\mathrm{i}\partial_t g_2 - 2L^2 g_2 - 2pLg_1 - \varepsilon |p|^2 g_2 + \partial_x p g_1 = 0
\end{aligned}\tag{8}$$

where $L \in \mathrm{L}(V)$ is a linear operator over V. For operators H and Λ such that

$$L^\dagger H - H\Lambda = 0$$

find an invertible operator Φ with

$$\begin{aligned}
&\partial_x \Phi = (g_1 \otimes g_1^\dagger - \varepsilon g_2 \otimes g_2^\dagger) H\\
&\mathrm{i}\partial_t \Phi = 2[\partial_x(\Phi\Lambda - L\Phi) + \varepsilon(pg_1 \otimes g_2^\dagger - p^* g_2 \otimes g_1^\dagger)H]\\
&L\Phi + \Phi\Lambda = (g_1 \otimes g_1^\dagger + \varepsilon g_2 \otimes g_2^\dagger) H\\
&H^\dagger \Phi = \Phi^\dagger H
\end{aligned}$$

where the first two are compatible equations and the second two restrict the possible initial conditions. Then,

$$\hat{p} = p + \langle g_1^\dagger, H\Phi^{-1} g_2\rangle$$

solves (7) and the following relation holds

$$|\hat{p}|^2 = |p|^2 + \varepsilon \partial_x^2 \ln\det\Phi.$$

If for a seed solution p we find L, Λ, H, g_i and Φ as prescribed in the above theorem and thus a new solution $\hat{p}$, one easily realizes that MLM^{-1}, $(M^\dagger)^{-1}\Lambda M^\dagger$, $(M^\dagger)^{-1} H M^\dagger$, Mg_i and $M\Phi M^\dagger$ with $M \in \mathrm{GL}(V)$, satisfies all the requirements of the theorem and moreover gives the same dressed solution $\hat{p}$ as with the previous data. Hence, there is an action of $\mathrm{GL}(V)$ that leaves invariant the Darboux transformation. This fact allows us to pick L in a canonical form, for example if $\dim V < \infty$ we can choose L in its Jordan canonical form. In the next section we shall analyse in more detail some solutions associated with these Jordan canonical forms for L.

For the second method we need to introduce a special class of linear operators $\mathcal{P}_\varepsilon$ on V,

$$\mathcal{P}_\varepsilon = \{J \in \mathrm{L}(V) : JJ^* + \varepsilon = 0\}.$$

Our construction stems from the following observation.

Proposition 4. (i) Given solutions g_1 and g_2 of (2) and $I \in \mathcal{P}_\varepsilon$ with the interlacing property $IL^* + LI = 0$ then

$$g_1 = Ig_2^*$$

imply $q = \varepsilon p^*$.

(ii) Given solutions γ_1 and γ_2 of (3) and $J \in \mathcal{P}_\varepsilon$ with the interlacing property $\Lambda^* J + J\Lambda = 0$ then

$$\gamma_2 = \gamma_1^* J$$

imply $q = \varepsilon p^*$.

Proof. We shall prove (i), the proof of (ii) goes analogously. Because the particular relation among g_1 and g_2 (2) can be written as

$$\begin{cases} I\partial_x g_2^* - LIg_2^* + \varepsilon qIg_1^* = 0 \\ \partial_x g_2 + Lg_2 + pg_1 = 0 \end{cases}$$

$$\begin{cases} I\mathrm{i}\partial_t g_2^* + 2L^2 Ig_2^* - \varepsilon 2qLIg_1^* + pqIg_2^* - \varepsilon\partial_x qIg_1^* = 0 \\ \mathrm{i}\partial_t g_2 - 2L^2 g_2 - 2pLg_1 - pqg_2 + \partial_x pg_1 = 0. \end{cases}$$

Using the interlacing property between L and I and comparing the equations in each set we see that we need $q = \varepsilon p^*$. □

Now, we proceed as in the previous method. We take a solution p of (7), thus $p, q = \varepsilon p^*$ solves (1), and we choose wavefunctions as in proposition 4 in order to perform a Darboux transformation. The Darboux operator Φ is characterized by

$$\partial_x\Phi = Ig_2^* \otimes \gamma_1 - g_2 \otimes \gamma_1^* J$$
$$\mathrm{i}\partial_t\Phi = 2[\partial_x(\Phi\Lambda - L\Phi) + pIg_2^* \otimes \gamma_1^* J - \varepsilon p^* g_2 \otimes \gamma_1]$$
$$L\Phi + \Phi\Lambda = Ig_2^* \otimes \gamma_1 + g_2 \otimes \gamma_1^* J.$$

But, the first two equations imply that $\partial_x(I^*\Phi + \Phi^* J) = 0$ and $\partial_t(I^*\Phi + \Phi^* J) = 0$, so that we can choose the initial condition such that

$$I^*\Phi + \Phi^* J = 0.$$

With such a Darboux operator we look to the transformed wavefunctions $\hat{g}_i$ and $\hat{\gamma}_i$, $i = 1, 2$:

$$\hat{g}_1 = \Phi^{-1}g_1 = \Phi^{-1}Ig_2^* = -J^*(\Phi^{-1})^* g_2^* = \hat{I}\hat{g}_2^*$$
$$\hat{\gamma}_2 = \gamma_2\Phi^{-1} = \gamma_1^* J\Phi^{-1} = -\gamma_1^*(\Phi^{-1})^* I^* = \hat{\gamma}_1^*\hat{J}.$$

Here $\hat{I} = -J^*$, $\hat{J} = -I^* \in \mathcal{P}_\varepsilon$. Now, proposition 4 implies that $\hat{q} = \varepsilon\hat{p}^*$, and therefore the Darboux transformation preserves the NLS reduction again.

The above results can be condensed in the following.

Theorem 3. Given a solution p of (7) and linear operators $L, \Lambda, I, J \in \mathrm{L}(V)$ on a complex linear space such that

$$II^* + \varepsilon = 0 \qquad JJ^* + \varepsilon = 0$$
$$LI + IL^* = 0 \qquad \Lambda^* J + J\Lambda = 0$$

choose a vector wavefunction g with values in V solving the Lax pair

$$\begin{aligned} &\partial_x g + Lg + pIg^* = 0 \\ &\mathrm{i}\partial_t g - 2L^2 g - 2pLIg^* - \varepsilon|p|^2 g - \partial_x pIg^* = 0 \end{aligned} \tag{9}$$

and a function γ taking values in V^* solving the adjoint Lax pair

$$\begin{aligned} &\partial_x\gamma - \gamma\Lambda - p\gamma^* J = 0 \\ &\mathrm{i}\partial_t\gamma - 2\gamma\Lambda^2 - 2p\gamma^* J\Lambda - \varepsilon|p|^2\gamma - \partial_x p\gamma^* J = 0. \end{aligned} \tag{10}$$

Find an invertible operator Φ solving the compatible equations

$$\partial_x\Phi = Ig^* \otimes \gamma - g \otimes \gamma^* J$$
$$\mathrm{i}\partial_t\Phi = 2[\partial_x(\Phi\Lambda - L\Phi) + pIg^* \otimes \gamma^* J - \varepsilon p^* g \otimes \gamma]$$

with initial conditions such that

$$L\Phi + \Phi\Lambda = Ig^* \otimes \gamma + g \otimes \gamma^* J$$
$$I^*\Phi + \Phi^* J = 0.$$

Then, the function

$$\hat{p} = p + 2\langle\gamma, \Phi^{-1} g\rangle$$

solves the NLS equation (7). Moreover,

$$|\hat{p}|^2 = |p|^2 + \varepsilon \partial_x^2 \ln \det \Phi.$$

Observe that in the focusing case $\varepsilon = 1$ there is no operator in $\mathcal{P}_1$ when $V = \mathbb{C}$. At least we need a two-dimensional space. We see that the linear systems (9) and (10) can be considered as new vector Lax pair for the NLS equation. The standard Lax pair for it is that given in theorem 2.

For $M, N \in \mathrm{GL}(V)$ the transformation

$$L, I, g, \Lambda, J, \gamma, \Phi \to MLM^{-1}, MI(M^*)^{-1}, Mg, N^{-1}\Lambda N, (N^*)^{-1}JN, \gamma N, M\Phi N$$

gives from data satisfying the conditions in the above theorem new data satisfying the same conditions. Moreover, the associated solution $\hat{p}$ is not modified by the transformation. Therefore, we have a double action of $\mathrm{GL}(V)$, right and left actions $(\mathrm{GL}(V)_{\mathrm{L}} \times \mathrm{GL}(V)_{\mathrm{R}})$ for which the Darboux transformation is invariant. This symmetry allows us to pick L and Λ in a canonical form, for example in the finite-dimensional case in its Jordan canonical forms.

4. Examples

This section is devoted to the analysis of some examples of the solutions of the NLS equation (7) obtained by applying theorems 2 and 3. As a seed solution of (7) we pick the 'constant' background

$$p(x, t) = \rho \exp(-2\mathrm{i}\varepsilon\rho^2 t) \tag{11}$$

with $\rho \in \mathbb{R}_+$ a non-negative real number. We recall the relevance of solutions with nonzero finite asymptotic values of the NLS equation [3]. We divide the section into two subsections, first we analyse the solutions given by the method stated in theorem 2, then we study the solutions associated with theorem 3.

4.1. First method

We apply theorem 2 to the seed solution given in (11). First we need to fix the algebraic data given by the linear operators L, Λ and H.

The operator H is related to the initial value Φ_0 of Φ, we shall choose $H = \mathrm{id}$ as the identity which implies that Φ is a Hermitian operator, $\Phi = \Phi^\dagger$. (Another possible choice is to fix Φ_0 as the identity and then H is where the parameters of the solution appear.)

With this choice, $H = \mathrm{id}$, we have $\Lambda = L^\dagger$. Then, the equations defining the wavefunctions g_1 and g_2 are

$$\begin{aligned}
&\partial_x g_1 - Lg_1 - \varepsilon\rho \exp(2\mathrm{i}\varepsilon\rho^2 t) g_2 = 0 \\
&\partial_x g_2 + Lg_2 + \rho \exp(-2\mathrm{i}\varepsilon\rho^2 t) g_1 = 0 \\
&\mathrm{i}\partial_t g_1 + 2L^2 g_1 + 2\varepsilon\rho \exp(2\mathrm{i}\varepsilon\rho^2 t) L g_2 + \varepsilon\rho^2 g_1 = 0 \\
&\mathrm{i}\partial_t g_2 - 2L^2 g_2 - 2\rho \exp(-2\mathrm{i}\varepsilon\rho^2 t) L g_1 - \varepsilon\rho^2 g_2 = 0.
\end{aligned}$$

The solution to this linear system is given by

$$g_1(x, t) = \exp(\mathrm{i}\varepsilon\rho^2 t) h_1(x, t) \qquad g_2(x, t) = \exp(-\mathrm{i}\varepsilon\rho^2 t) h_2(x, t)$$

where

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \exp(\mathcal{S}) \begin{pmatrix} g_1(0) \\ g_2(0) \end{pmatrix}$$

with $\mathcal{S} := \mathcal{H}\mathcal{L}$ an operator in $V \oplus V$, where the operators $\mathcal{H}$ and $\mathcal{L}$ can be written in block form as

$$\mathcal{H}(x,t) := \operatorname{diag}(x + 2\mathrm{i}Lt, x + 2\mathrm{i}Lt)$$

$$\mathcal{L} := \begin{pmatrix} L & \varepsilon\rho \\ -\rho & -L \end{pmatrix}.$$

Observe that $\mathcal{M} := \mathcal{L}^2 = \operatorname{diag}(L^2 - \varepsilon\rho^2, L^2 - \varepsilon\rho^2)$ so that

$$\exp(\mathcal{S}) = \sum_{n\geqslant 0} \frac{\mathcal{H}^{2n}}{(2n)!}\mathcal{M}^n + \sum_{n\geqslant 0} \frac{\mathcal{H}^{2n+1}}{(2n+1)!}\mathcal{M}^n\mathcal{L}.$$

In the finite-dimensional case, without lack of generality, we can take L to be in its Jordan form: $L = J_{\ell_1} \oplus \ldots \oplus J_{\ell_m}$, where ℓ_j are the eigenvalues of L and J_{ℓ_j} is a Jordan block corresponding to that eigenvalue. One can show that the operator $L^2 - \varepsilon\rho^2$ has a square root if $\ell_j^2 - \varepsilon\rho^2 \neq 0$, for all j, if so the operator is invertible as well. This will be called the generic case, that we shall analyse with certain detail. For the non-generic case one has rational solutions of NLS and we shall present here only the simplest case.

4.1.1. Generic case. If the operator $L^2 - \varepsilon\rho^2$ has a square root, say K, such that

$$K^2 = L^2 - \varepsilon\rho^2$$

we introduce the operators

$$R := \varepsilon(K - L)/\rho = -\rho(K + L)^{-1}$$

$$\Psi(x,t) := \exp(K(x + 2\mathrm{i}Lt)).$$

In terms of these quantities the general solution can be written as

$$h_1(x,t) = \Psi(x,t)^{-1} f_1(x,t)$$

$$h_2(x,t) = \Psi(x,t)^{-1} f_2(x,t)$$

where

$$f_1(x,t) := G(x,t) + Rv_2$$

$$f_2(x,t) := RG(x,t) + \varepsilon v_2$$

with

$$G(x,t) := \exp(2K(x + 2\mathrm{i}Lt))v_1$$

being $v_1, v_2 \in V$ arbitrary vectors related to the initial conditions for the wavefunctions. Namely, $g_1(0) = v_1 + Rv_2$ and $g_2(0) = Rv_1 + \varepsilon v_2$.

Now, we introduce

$$\varphi = \Psi\Phi\Psi^\dagger$$

and the new solution of the NLS equation is

$$p = \exp(-2\mathrm{i}\varepsilon\rho^2 t)(\rho + \langle f_1^\dagger, \varphi^{-1} f_2\rangle).$$

Moreover, $\partial_x^2 \ln\det\varphi = \partial_x^2 \ln\det\Phi$.

All these results can be gathered together in the following.

Proposition 5. Let ρ be a non-negative real number and $L \in \mathrm{L}(V)$ a linear operator in a complex linear space V such that the operator $L^2 - \varepsilon\rho^2$ has a square root, say K:

$$K^2 = L^2 - \varepsilon\rho^2.$$

Let $R = \varepsilon(K - L)/\rho$ and

$$f_1(x,t) := G(x,t) + Rv_2$$
$$f_2(x,t) := RG(x,t) + \varepsilon v_2$$

with

$$G(x,t) = \exp(2K(x + 2\mathrm{i}Lt))v_1$$

and $v_1, v_2 \in V$. Choose an invertible operator φ subject to the following compatible equations

$$\partial_x\varphi = K\varphi + \varphi K^\dagger + f_1 \otimes f_1^\dagger - \varepsilon f_2 \otimes f_2^\dagger$$
$$\mathrm{i}\partial_t\varphi = (\partial_x\varphi - K\varphi)L^\dagger - L(\partial_x\varphi - \varphi K^\dagger) + \rho(\varepsilon f_1 \otimes f_2^\dagger - f_2 \otimes f_1^\dagger)$$
$$L\varphi + \varphi L^\dagger = f_1 \otimes f_1^\dagger + \varepsilon f_2 \otimes f_2^\dagger.$$

Then we can construct solutions of (7) as follows

$$p(x,t) = \exp(-2\mathrm{i}\varepsilon\rho^2 t)(\rho + \langle f_1^\dagger, \varphi^{-1} f_2\rangle)$$

for which $|p|^2 = \rho^2 + \varepsilon\partial_x^2 \ln\det\varphi$ holds.

In the finite-dimensional case one takes $V = \mathbb{C}^n$, then by a change of basis we could write the matrix corresponding to L in its Jordan canonical form. We first suppose that $L = \mathrm{diag}(\ell_1, \ldots, \ell_n)$ is diagonal with eigenvalues $\ell_j \in \mathbb{C}$. Later we shall consider a non-diagonal case associated with the simplest Jordan block.

Diagonal case. Now, we have

$$K = \mathrm{diag}(k_1, \ldots, k_n) \qquad R = \mathrm{diag}(R_1, \ldots, R_n)$$

and

$$G = (G_1, \ldots, G_n)^t$$

with k_j a square root of $\ell_j^2 - \varepsilon\rho^2$, $R_j = \varepsilon(k_j - \ell_j)/\rho$ and $G_j(x,t) = G_{j0}\exp(2k_j(x + 2\mathrm{i}\ell_j t))$ with $G_{j0} = v_{1j} \in \mathbb{C}$ an arbitrary complex number. In this case we can choose generically v_2 as the vector $(1, \ldots, 1)^t$. For the components of the vectors f we have

$$f_{1j}(x,t) = G_j(x,t) + R_j$$
$$f_{2j}(x,t) = R_j G_j(x,t) + \varepsilon.$$

The Darboux operator φ satisfies $L\varphi + \varphi L^\dagger = f_1 \otimes f_1^\dagger + \varepsilon f_2 \otimes f_2^\dagger$ which can be written, for non-vanishing $\ell_i + \ell_j^*$, in matrix form as

$$\begin{aligned}\varphi_{ij} &= \frac{1}{\ell_i + \ell_j^*}(f_{1i} f_{1j}^* + \varepsilon f_{2i} f_{2j}^*) \\ &= \frac{1}{\ell_i + \ell_j^*}[(G_i + R_i)(G_j + R_j)^* + \varepsilon(R_i G_i + \varepsilon)(R_j G_j + \varepsilon)^*].\end{aligned}$$

The new solution p of NLS is

$$p = \exp(-2\mathrm{i}\varepsilon\rho^2 t)\left(\rho + 2\frac{\sum_{j=1}^n (R_j G_j + \varepsilon)\det\varphi_j}{\det\varphi}\right)$$

where φ_j is the matrix obtained by replacing in φ the jth row by $(f_{11}^*, \ldots, f_{1n}^*)$.

When $\rho = 0$ this corresponds, in the anomalous dispersion case $\varepsilon = 1$, to the nth bright soliton solution [14, 6]. In that case

$$
\begin{aligned}
R_j &= 0 \\
G_j(x,t) &= G_{j0}\exp(2\ell_j(x + 2\mathrm{i}\ell_j t)) \\
f_{1j} &= G_j \\
f_{2j} &= 1 \\
\varphi_{ij} &= (\varepsilon + G_i G_j^*)/(\ell_i + \ell_j^*)
\end{aligned}
$$

and the formulae above for p are those for the nth bright soliton solution ($\varepsilon = 1$) as appear in [6, (5.34), (5.37), (5.38)], see also the seminal paper [14].

Hence, the above solution can be considered in the focusing case as a deformation of the bright soliton depending upon the parameter ρ. For a more detailed understanding we particularize to the $n = 1$ case: the deformation of the 1-soliton.

Deformation of the bright 1-soliton: topological features. First we study the relation $\ell^2 - k^2 = \varepsilon\rho^2$ among the complex parameters $\ell, k \in \mathbb{C}$. For a complex number z we shall denote by $z_R := \operatorname{Re} z$ and $z_I = \operatorname{Im} z$ its real and imaginary parts, respectively. Then, the relation between ℓ and k imply

$$
\begin{aligned}
&k_I = \alpha\ell_R \qquad \ell_I = \alpha k_R \\
&\alpha^2 = \frac{\sqrt{(\ell_R^2 - \ell_I^2 - \varepsilon\rho^2)^2 + 4\ell_I^2\ell_R^2} - (\ell_R^2 - \ell_I^2 - \varepsilon\rho^2)}{2\ell_R^2}.
\end{aligned}
$$

Now, we write

$$G(x,t) = G_0\exp(2\mathrm{i}\alpha\ell_R(x - ut))\exp(2\ell_I(x - vt)/\alpha)$$

here we suppose that $k_R k_I \neq 0$ and $v = 2(\ell_I + (k_I/k_R)\ell_R) = 2(\ell_I + \alpha^2\ell_R^2/\ell_I)$ and $u = 2(\ell_I - (k_R/k_I)\ell_R) = 2(1 - 1/\alpha^2)\ell_I$. We have

$$
\begin{aligned}
\varphi &= \frac{1 + \varepsilon|R|^2}{2\ell_R}\left(\varepsilon + \frac{4R_R}{1 + \varepsilon|R|^2}\operatorname{Re} G + |G|^2\right) \\
&= \frac{\alpha}{\alpha\ell_R + \ell_I}\left(\varepsilon + \frac{2\rho}{(1 + \alpha^2)\ell_R}|G_0|\cos(2\alpha\ell_R(x - ut) + \delta)\exp(2\ell_I(x - vt)/\alpha)\right. \\
&\quad \left. + |G_0|^2\exp(4\ell_I(x - vt)/\alpha)\right)
\end{aligned}
$$

where $\delta = \operatorname{Arg} G_0$ is the argument of G_0. The amplitude reads

$$p = \exp(-2\mathrm{i}\varepsilon\rho^2 t)(\rho + 2(G + R)^*(\varepsilon + GR)/\varphi).$$

The explicit expression of φ was already known [11], however, a relevant property of such a solution is its topological nature, that apparently was missed in [11], and which we shall discuss here. We assume that $k_R > 0$, then using the above formula and the asymptotic behaviour of G, after some computations one can show that

$$
\begin{aligned}
p &\sim -\rho\exp(-2\mathrm{i}\rho^2 t)C(\mathrm{i}\alpha) \qquad x \to -\infty \\
p &\sim -\rho\exp(-2\mathrm{i}\rho^2 t)C(-\mathrm{i}\alpha) \qquad x \to \infty
\end{aligned}
$$

where

$$C(x) := \frac{1 + x}{1 - x}$$

is the Cayley transform of x.

The asymptotic value of the modulus of the amplitude is $|p| \sim \rho$, $x \to \pm\infty$, that follows immediately from $|p|^2 = \rho^2 + \varepsilon\partial_x^2 \ln\varphi$. For the argument $\operatorname{Arg}\hat{p}$ the situation is different, as $\operatorname{Arg}((1+\mathrm{i}\alpha)/(1-\mathrm{i}\alpha)) = 2\operatorname{Arg}(1+\mathrm{i}\alpha) = 2\arctan\alpha$ we conclude that

$$\begin{aligned} \operatorname{Arg} p &\sim \pi - 2\rho^2 t + 2\arctan\alpha =: \phi_- \qquad x \to -\infty \\ \operatorname{Arg} p &\sim \pi - 2\rho^2 t - 2\arctan\alpha =: \phi_+ \qquad x \to \infty \end{aligned}$$

and therefore the difference among the asymptotic arguments of the amplitude at $x = -\infty$ and $x = \infty$ is for any t

$$\Delta\phi := \phi_+ - \phi_- = -4\arctan\alpha.$$

We see that unless $k + \ell \in \mathbb{R}$ ($R \in \mathbb{R}$) our solution is of a topological character. But $k^2 - \ell^2 = -\rho^2 \in \mathbb{R}$ so that $k + \ell$ real implies $k - \ell$ real as well, that is k and ℓ real. Therefore, unless $\ell \in \mathbb{R}$ and $\ell^2 > \rho^2$, we are dealing with solutions of a topological nature as the dark solitons of the NLS equation in the standard dispersion regime [9], $\varepsilon = -1$.

In the standard dispersion case, $\varepsilon = -1$, the solution is singular. But when deforming the bright 1-soliton, $\varepsilon = 1$, we have a solitonic exponentially localized perturbation over the background that moves with velocity v, and that amplitude is modulated by plane wave with velocity u.

In figure 1 we plot a generic topological deformation of the bright 1-soliton of the focusing NLS equation.

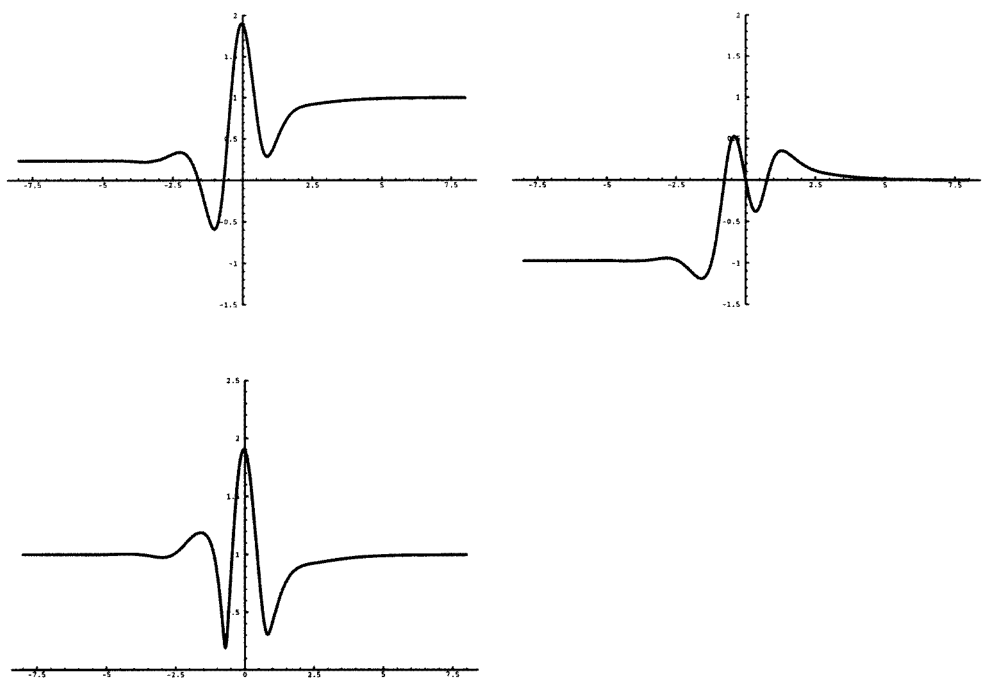

Figure 1. The topological deformation of the bright 1-soliton solution of the NLS for anomalous dispersion. Our data is $\ell_R = \ell_I = \rho = |G_0| = 1$ and $\delta = 0$. The plots are taken for $t = 0$ and in a co-moving frame of velocity $-v$. The two first graphs show the real and imaginary parts of the amplitude while the last one is the modulus of the amplitude. Observe the topological character of the solution in the two first plots. As this solution evolves in t a localized periodic modulation appears around $x = 0$ with period $T = \pi/(\alpha\ell_R(u - v))$.

Homoclinic orbits. For $\varepsilon = 1$ and $k_R = 0$ the behaviour of the solution is rather different from the one described above. For the non-trivial case $\ell \in \mathbb{R}$, $0 < \ell_R < \rho$, one has $\alpha = \sqrt{\rho^2 - \ell_R^2}/\ell_R$. We introduce $P := 2\sqrt{\rho^2 - \ell_R^2}$ in terms of which one writes $\Omega = -P\sqrt{4\rho^2 - P^2} = -4\ell_R\sqrt{\rho^2 - \ell_R^2}$, $\sin\phi = P/2\rho$, and also $|G_0|\ell_R/\rho = \exp\gamma$ so that $|G_0|^2 = A_{12}\exp(2\gamma)$ with $A_{12} = \rho^2/\ell_R^2 = \sec^2\phi$. Then, the function φ reads

$$\varphi(x,t) = 1 + 2\exp(\Omega t + \gamma)\cos(2Px + \delta) + A_{12}\exp(2\Omega t + 2\gamma).$$

The associated amplitude p corresponds to the single homoclinic orbit of the focusing NLS, see formula (3.3.9) in [2] and references therein.

Obviously, a combination of homoclinic orbits can be constructed from the n-dimensional case discussed above by choosing $\ell_{jI} = 0$ and $0 < \ell_{jR} < \rho$.

A simple Jordan block. We give the results corresponding to the simplest non-diagonalizable operator L. Assume $V = \mathbb{C}^2$ and let L be

$$L = \begin{pmatrix} \ell & 1 \\ 0 & \ell \end{pmatrix}$$

a Jordan block with eigenvalue $\ell \in \mathbb{C}$. Let $k \in \mathbb{C}^\times$ be a non-vanishing root of $k^2 = \ell^2 - \varepsilon\rho^2$ and define

$$\begin{aligned} \eta(x,t) &:= 2k(x + 2\mathrm{i}\ell t) \\ P(x,t) &:= v_{11} + 2v_{12}(\ell/kx + 2\mathrm{i}(k + \ell^2/k)t) \\ r &:= \frac{\varepsilon(k-\ell)}{\rho} \end{aligned}$$

then the corresponding modulus of the amplitude (for $\ell_R \neq 0$) is given by the expression

$$|p|^2 = \rho^2 + \partial_x^2 \ln(|E|^2 + \varepsilon F^2)$$

with

$$\begin{aligned} E(x,t) &:= r/kv_{12}^2\exp(2\eta) + (\varepsilon - r^2)(v_{22}P - v_{12}v_{21})\exp(\eta) - \varepsilon r/kv_{22}^2 \\ F(x,t) &:= ((1 + \varepsilon|r|^2)(|v_{12}|^2\exp(2\eta_R) + \varepsilon|v_{22}|^2) \\ &\quad + 4r_R\,\mathrm{Re}(v_{22}^* v_{12}\exp(\mathrm{i}\eta_I))\exp(\eta_R))/(2\ell_R). \end{aligned}$$

Here $v_{ij} \in \mathbb{C}$ are arbitrary constants.

This solution is constructed in terms of quasi-exponentials, not solely exponentials. Therefore, it could be considered as a rational deformation of the topological deformation of the bright soliton. If one takes $L = \begin{pmatrix} \ell_1 & 1 \\ 0 & \ell_2 \end{pmatrix}$ one obtains the same solution that for $L = \mathrm{diag}(\ell_1, \ell_2)$ if $\ell_1 \neq \ell_2$: for the focusing case the topological deformation is of the bright 2-soliton solution. The limit $\ell_1 \to \ell_2$ is the one precisely giving our rational deformation. But this statement is clear from the form of the operator L; however, at the level of the explicit expression of the solution it is not clear how it can be done. In any case it could be considered as a rational degeneration of the topological deformation of the bright 2-soliton solution (focusing case).

4.1.2. Non-generic case. The limit commented in the previous paragraph can be taken to be $\ell_1, \ell_2 \to \sqrt{\varepsilon}\rho$, which is not contained in the previous discussion because then $k = 0$ and there is no square root available. We shall study this particular case here by using the general form of the wavefunctions presented at the beginning of this section. Now,

$$L = \begin{pmatrix} \sqrt{\varepsilon}\rho & 1 \\ 0 & \sqrt{\varepsilon}\rho \end{pmatrix}.$$

We introduce the following notation

$$\eta(x,t)L = x + 2\mathrm{i}\sqrt{\varepsilon}t.$$

In the focusing case, $\varepsilon = 1$, the square modulus of the amplitude is

$$|p|^2 = \rho^2 + \partial_x^2 \ln(|E|^2 + F^2)$$

where

$$\begin{aligned} E &:= v_{11}v_{22} - v_{12}v_{21} + 2\eta v_{12}v_{22} - \rho\eta^2(v_{12}^2 - v_{22}^2) + (v_{12} + v_{22})^2 Q \\ F(x,t) &:= |v_{12}|^2 + |v_{22}|^2 + 2\rho x(|v_{12}|^2 - |v_{22}|^2) + 4\rho t\,\mathrm{Im}(v_{12}^* v_{22}) \\ &\quad + 2\rho^2(x^2 + 4\rho^2 t^2)|v_{12} + v_{22}|^2 \end{aligned}$$

with $Q(x,t) := 2(\mathrm{i}\rho t - \rho^2\eta^3/3)$. Here $v_{ij} \in \mathbb{C}$. This is a rational solution of the focusing NLS equation.

4.2. Second method

We apply theorem 3 to the seed solution given in (11).

The equations defining the function g are

$$\begin{aligned} &\partial_x g + Lg + \rho\exp(-2\mathrm{i}\varepsilon\rho^2 t)Ig^* = 0 \\ &\mathrm{i}\partial_t g - 2L^2 g - 2\rho\exp(-2\mathrm{i}\varepsilon\rho^2 t)LIg^* - \varepsilon\rho^2 g = 0 \end{aligned}$$

whose solution is

$$g(x,t) = \exp(-\mathrm{i}\varepsilon\rho^2 t)h(x,t)$$

with

$$\begin{pmatrix} h \\ h^* \end{pmatrix} = \exp(-\mathcal{S})\begin{pmatrix} v \\ v^* \end{pmatrix}.$$

Here $\mathcal{S} := \mathcal{H}\mathcal{L}$ is an operator in $V \oplus V$, where the operators $\mathcal{H}$ and $\mathcal{L}$ can be written in block form as

$$\begin{aligned} &\mathcal{H}(x,t) := \mathrm{diag}(\eta, \eta^*) \\ &\mathcal{L} := \begin{pmatrix} L & \rho I \\ \rho I^* & L^* \end{pmatrix} \end{aligned}$$

with $\eta(x,t) := x + 2\mathrm{i}Lt$. Observe that $\mathcal{M} := \mathcal{L}^2 = \mathrm{diag}(L^2 - \varepsilon\rho^2, (L^*)^2 - \varepsilon\rho^2)$ so that

$$\exp(\mathcal{S}) = \sum_{n\geqslant 0} \frac{\mathcal{H}^{2n}}{(2n)!}\mathcal{M}^n + \sum_{n\geqslant 0} \frac{\mathcal{H}^{2n+1}}{(2n+1)!}\mathcal{M}^n\mathcal{L}.$$

When $L^2 - \varepsilon\rho^2$ has an invertible square root K the above expansion simplifies to give

$$h = \exp(K\eta)w_+ + \exp(-K\eta)w$$

with

$$
\begin{aligned}
w_+ &= K^{-1}[(K-L)v - \rho I v^*]/2 \\
w_- &= K^{-1}[(K+L)v + \rho I v^*]/2.
\end{aligned}
\tag{12}
$$

The relation $LI + IL^* = 0$ implies $F(L)I = IF(-L^*)$ for any function F defined by a power series. For example, $KI = IK^*$ and $\exp(K\eta)I = I\exp(K^*\eta^*)$. The relations (12) are equivalent to the following equations

$$(L \pm K)w_\pm + \rho I w_\pm^* = 0.$$

If Λ is such that there exists an invertible operator, say κ, solving $\kappa^2 = \Lambda^2 - \varepsilon\rho^2$ a similar approach leads to a wavefunction $\gamma = \exp(-\mathrm{i}\varepsilon\rho^2 t)\delta$ with

$$\delta = \omega_+ \exp(\kappa\xi) + \omega_- \exp(-\kappa\xi)$$

where $\xi := x - 2\mathrm{i}\Lambda t$, and the ω's are linear functionals characterized by

$$\omega_\pm(\Lambda \mp \kappa) + \rho\omega_\pm^* J = 0.$$

As we did with the first method we shall factor out from the wavefunctions an exponential contribution. Namely:

$$
\begin{aligned}
h &=: \exp(K\eta) f \\
\delta &=: \beta \exp(\kappa\xi)
\end{aligned}
$$

and introduce

$$\varphi := \exp(-K\eta)\Phi\exp(-\kappa\xi).$$

These definitions together with theorem 3 leads us to:

Proposition 6. Let $\rho \geqslant 0$ and $L, \Lambda, I, J \in \mathrm{L}(V)$ be linear operators on the complex linear space V subject to

$$
\begin{aligned}
II^* + \varepsilon &= 0 \qquad & JJ^* + \varepsilon &= 0 \\
LI + IL^* &= 0 \qquad & \Lambda^* J + J\Lambda &= 0
\end{aligned}
$$

and such that there exists invertible operators K, κ with

$$K^2 = L^2 - \varepsilon\rho^2 \qquad \kappa^2 = \Lambda^2 - \varepsilon\rho^2.$$

Using the notation

$$
\begin{aligned}
\eta(x,t) &:= x + 2\mathrm{i}Lt \\
\xi(x,t) &:= x - 2\mathrm{i}\Lambda t
\end{aligned}
$$

we define the wavefunctions

$$
\begin{aligned}
f &:= w_+ + \exp(-2K\eta)w_+ \\
\beta &:= \omega_+ + \omega_- \exp(-2\kappa\xi)
\end{aligned}
$$

with $w_\pm \in V$ and $\omega_\pm \in V^*$ satisfying

$$
\begin{aligned}
&(L \pm K)w_\pm + \rho I w_\pm^* = 0 \\
&\omega_\pm(\Lambda \mp \kappa) + \rho\omega_\pm^* J = 0.
\end{aligned}
$$

We introduce an invertible operator solution, say φ, of the following compatible equations

$$
\begin{aligned}
&\partial_x\varphi = If^* \otimes \beta - f \otimes \beta^* J - K\varphi - \varphi\kappa \\
&\mathrm{i}\partial_t\varphi = 2[(\partial_x\varphi + K\varphi)\Lambda - L(\partial_x\varphi + \varphi\kappa)] + \rho(If^* \otimes \beta^* J - \varepsilon f \otimes \beta) \\
&L\varphi + \varphi\Lambda = If^* \otimes \beta - f \otimes \beta^* J \\
&I^*\varphi + \varphi^* J = 0.
\end{aligned}
$$

Then,

$$p = \exp(-2\mathrm{i}\varepsilon\rho^2 t)(\rho + 2\langle\beta, \varphi^{-1} f\rangle)$$

solves (7). Moreover,

$$|p|^2 = \rho^2 + \varepsilon\partial_x^2 \ln(\det\varphi)$$

Within this proposition one finds large families of solutions of the NLS equation. For example the dark solitons [15] of the defocusing case appear in this scheme. This is a relevant fact: for the first time the dark solitons have been derived with the use of Darboux transformations.

The standard dispersion NLS corresponds to $\varepsilon = -1$, then one can choose $I = J = \mathrm{I}_n$, $L = \mathrm{i}\,\mathrm{diag}(\ell_1, \ldots, \ell_n)$ and $\Lambda = \mathrm{i}\,\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, with $-\rho < \ell_j, \lambda_j < \rho$, $K = \mathrm{diag}(k_1, \ldots, k_n)$ and $\kappa = \mathrm{diag}(\kappa_1, \ldots, \kappa_n)$ with $k_j = \sqrt{\rho^2 - \ell_j^2}$ and $\kappa_j = \sqrt{\rho^2 - \lambda_j^2}$. If $\alpha_j := \arctan(\ell_j/k_j)$ and $\theta_j := \arctan(\lambda_j/\kappa_j)$ we have

$$\begin{aligned} w_{+,j} &= \mathrm{i}a_{+,j} \exp(-\mathrm{i}\alpha_j/2) \qquad & w_{-,j} &= a_{-,j} \exp(\mathrm{i}\alpha_j/2) \\ \omega_{+,j} &= b_{+,j} \exp(\mathrm{i}\theta_j/2) \qquad & \omega_{-,j} &= \mathrm{i}b_{-,j} \exp(-\mathrm{i}\theta_j/2) \end{aligned}$$

with $a_{\pm,j}, b_{\pm,j} \geqslant 0$, non-negative real numbers.

In the generic case we can factor out the $w_{+,j}$ and the $\omega_{+,j}$. Hence if $f = (f_1, \ldots, f_n)^t$ and $\beta = (\beta_1, \ldots, \beta_n)$ are defined by

$$\begin{aligned} f_j &:= 1 - \mathrm{i}A_j \exp(-2k_j\eta_j + \mathrm{i}\alpha_j) \\ \beta_j &:= 1 + \mathrm{i}B_j \exp(-2\kappa_j\xi_j - \mathrm{i}\theta_j) \end{aligned}$$

where $\eta_j = x - 2\ell_j t$ and $\xi_j = x + 2\lambda_j t$ and $A_j, B_j \geqslant 0$, and the matrix φ has as entries

$$\varphi_{ij} = \frac{1}{\ell_i + \lambda_j} \mathrm{Re}((1 + \mathrm{i}A_i \exp(-2k_i\eta_i - \mathrm{i}\alpha_i))(1 + \mathrm{i}B_j \exp(-2\kappa_j\xi_j + \mathrm{i}\theta_j)))$$

the function

$$p = \exp(2\mathrm{i}\rho^2 t)(\rho + \mathrm{i}\langle\beta, \varphi^{-1} f\rangle)$$

is a solution of the defocusing NLS. This solution represents two sets of n dark soliton with velocities $4k_j\ell_j$ and $-4\kappa_j\lambda_j$, phases $\pi - \alpha_j$ and $\pi - \theta_j$ and centres related with $(\ln A_j)/K_j$ and $(\ln B_j)/\kappa_j$. The one dark soliton can be readily obtained when $n = 1$ and $B_1 = 0$. Observe that the formulae for multiple dark solitons of [6] have a different appearance. Our expressions, that give the same solutions, are closer to the standard formulae for the bright solitons and its topological deformations.

We conclude remarking that the solutions discussed within the first method: bright solitons and its topological and rational deformations can be reobtained here. For example one can take $V = W \oplus W$ with $W = \mathbb{C}^n$ and $I = -J = \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}$, but we shall go no further in this paper.

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